On the Number and Placement of Actuators for Independent Modal Space Control

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A new formulation of independent modal space control is developed to handle the attitude and shape control problem for large flexible spacecraft. The main advantage of this method is that one can obtain an analytical solution for the optimal control law for very high dimensional systems. The fundamental limitation of previous work—the requirement of one actuator for each mode to be controlled—is relaxed in the new formulation. The closed-loop design is obtained while independently assuring stability and the design may be iterated to improve closed-loop performance. The process is shown to be simple and efficient in a realistic numerical example of spacecraft shape and attitude control. The ease of control law generation by this approach is seen to be obtained at the expense of the ability to adjust directly the penalties on the actuator effort. Actuator placement is seen to be of fundamental importance, and methods are developed which are comparatively simple to use and which can determine optimal actuator locations.

Introduction

A NEW control design technique, independent modal space control (IMSC), has recently been developed as a method of generating control laws for large-dimensional harmonic systems such as future large flexible spacecraft.[1-4] In this paper, a simple and succinct summary of this method for optimal control problems is first presented, together with a balanced view of the pros and cons of the approach. The main advantage for linear-quadratic problems is that it allows one to solve the Riccati equation as a set of 2×2 Riccati equations, and thus allows one to obtain solutions for very large systems. However, in order to implement the method, one must use one actuator for each mode to be controlled, a fact which totally nullifies the stated advantage. A new formulation of independent modal space control is developed herein, which relaxes this restriction and allows one to use a reduced number of actuators by synthesizing an approximation to the optimal feedback, which is itself optimal with respect to a modified cost functional.

Another characteristic which distinguishes this method from the standard linear-quadratic design approach is that the control laws are generated without regard to the locations of the actuators. This result has led to the claim that the actuator locations are "immaterial" when IMSC is used.[4] A premise of this work is that actuator placement is of *primary* importance in the control of distributed parameter systems; therefore, the above statement warrants careful consideration. It is found here that the actuator locations do influence the ultimate physical realization of the control laws as generated by IMSC and, therefore, must be carefully chosen to assure a physically realizable control (i.e., within the physical limits of the control hardware). Methods for actuator placement optimization are developed for use in conjunction with IMSC, and the resulting actuator placement task may be decoupled from the control law design task (a feature unique to IMSC). Finally, the problem of control spillover is considered in the context of IMSC.

Summary and Discussion of IMSC

In this section we present a concise statement of the IMSC method distilled from Refs. 1-4 and compare the method to the more standard linear-quadratic approach. Optimal control of a distributed parameter system via IMSC is accomplished by first approximating the partial differential equation model by a finite set of ordinary differential equations, which for Refs. 3 and 4 take the form

$$M\ddot{q}(t)+Kq(t)=F(t);\qquad M>0,\qquad k>0 \tag{1}$$

where $F(t)$ is a generalized force vector. References 1 and 2 apply IMSC to undamped gyroscopically coupled systems. The approach will be generalized here to handle damped nongyroscopic systems, including rigid body modes.

A transformation matrix P satisfying $P^TMP=I$ and $P^TKP=\Omega=\mathrm{diag}(\omega_1^2,...,\omega_n^2)$ produces the *modal space representation*

$$\ddot{\nu}_r(t)+\omega_r^2\nu_r(t)=f_r(t)\qquad r=1,2,...,n \tag{2}$$

$$\nu(t)=[\nu_1(t)\cdots\nu_n(t)]^T=P^Tq(t) \tag{3}$$

$$f(t)=[f_1(t)\cdots f_n(t)]^T=P^TF(t) \tag{4}$$

A corresponding *state space representation* of the system is:

$$\dot{x}(t)=Ax(t)+W(t) \tag{5}$$

$$x(t)=[x_1(t)\cdots x_{2n}(t)]^T \tag{6}$$

$$x_{2r-1}(t)=\nu_r(t);\qquad x_{2r}(t)=\dot{\nu}_r(t)/\omega_r \tag{7}$$

$$W(t)=[W_1^T(t)\cdots W_n^T(t)]^T \tag{8}$$

$$W_r(t)=[0\,f_r(t)/\omega_r]^T \tag{9}$$

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$$A=\text{diag}(A_1,\ldots,A_n); \qquad A_r=\begin{bmatrix} 0 & \omega_r \\ -\omega_r & 0 \end{bmatrix} \tag{10}$$

This representation can be put into a more standard form,

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{11}$$

by expressing the driving term in Eq. (5) in terms of the actual control inputs $u(t)$ rather than the generalized forces $W(t)$. B takes the form

$$B=[B_1^T\cdots B_n^T]^T \tag{12}$$

$$B_r=\begin{bmatrix} 0 & \cdots & 0 \\ \phi_r(p_1)/\omega_r & \cdots & \phi_r(p_m)/\omega_r \end{bmatrix} \tag{13}$$

where $\phi_r(p_i)$ is the rth mode-shape function evaluated at position p_i when the associated actuator is a force actuator, and the spatial derivative of that mode-shape function in the case of a torque actuator.

The standard linear-quadratic result uses Eq. (11) and the cost functional

$$J=\int_0^T(x^TQx+u^TRu)\,\mathrm{d}t; \qquad Q\geq 0, \quad R>0 \tag{14}$$

to obtain

$$u(t)=-R^{-1}B^TK(t)x(t) \tag{15}$$

$$\dot{K}(t)=-KA-A^TK+KBR^{-1}B^TK-Q \tag{16}$$

It is apparent from Eq. (16) that the actuator location information, inherent in B, influences the optimal control law, Eq. (15).

In contrast, IMSC uses Eq. (5) and an alternative cost functional

$$J_{\text{IMSC}}=\int_0^T(x^TQ_{\text{IMSC}}x+W^TR_{\text{IMSC}}W)\,\mathrm{d}t \tag{17}$$

$$Q_{\text{IMSC}}=\text{diag}(Q_1,\ldots,Q_n) \qquad R_{\text{IMSC}}=\text{diag}(R_1,\ldots,R_n) \tag{18}$$

with Q_r and R_r being 2×2 matrices associated with each mode. By defining $w_r=[x_{2r-1}\ x_{2r}]^T$, the problem can be decomposed into a set of n second-order decoupled optimal control problems as follows.

$$J_{\text{IMSC}}=\sum_{r=1}^{n}J_r$$

$$=\sum_{r=1}^{n}\int_0^T(w_r^TQ_rw_r+W_r^TR_rW_r)\,\mathrm{d}t \tag{19}$$

$$W_r(t)=-R_r^{-1}K_r(t)w_r(t) \tag{20}$$

$$\dot{K}_r(t)=-K_rA_r-A_r^TK_r+K_rR_r^{-1}K_r-Q_r \tag{21}$$

Here, the generalized controls $W_r(t)$ have been determined independently, and in the modal space, hence the name independent modal space control. It remains to determine actuator commands u_i that can realize these generalized forces. We will address this later.

Since the feedback solution, Eq. (20), must yield a generalized force vector of the form specified in Eq. (9), a further restriction on R_{IMSC} is imposed. Expanding the second term in the integral of Eq. (19) gives $W_r^TR_rW_r=(f_r(t)/\omega_r)^2[R_r]_{22}$, from which we can see that only the element $[R_r]_{22}$ influences the cost functional. The form of $W_r(t)$ in Eq. (9) can be obtained by requiring the first row of R_r^{-1} of Eq. (20) to be zero, and by symmetry the remaining off-diagonal element is zero. If this is written $R_r^{-1}=\text{diag}(0,\ 1/\rho_r)$, then the required form for the original weighting matrix is

$$R_r=\text{diag}(\infty,\rho_r) \tag{22}$$

(The alternative of requiring the first row of $R_r^{-1}K_r$ to be zero can allow more freedom in R_r at the expense of freedom in Q_r.)

The above summary establishes the foundation for a comparison of the IMSC and standard linear-quadratic approaches. This is given in the following five remarks.

Remark 1: The IMSC approach has a significantly smaller computational requirement, since it requires the solution of n-decoupled 2×2 Riccati equations, Eq. (21), rather than a single $(2n\times 2n)$ Riccati equation, Eq. (16). In fact, an analytical solution of the 2×2 Riccati equation is given in Ref. 1 for the special case of an infinite time problem, Q_r the identity matrix, and A_r as in Eq. (10).

We can generalize this result to handle arbitrary Q_r and A_r matrices, so that modal damping and rigid body modes can be included in the system equation, Eq. (1). Without loss of generality we can choose the state vector to produce the companion form for A_r, since a coordinate transformation is easily accounted for in the choice of Q_r. No transformation of R_r is required, since it weights a scalar generalized control function $f_r(t)$. Then the elements of K_r are given in terms of roots of quadratic equations below:

$$A_r=\begin{bmatrix} 0 & 1 \\ -a_{21} & -a_{22} \end{bmatrix} \tag{23}$$

$$k_{12}^2+2a_{21}\rho_rk_{12}-q_{11}\rho_r=0 \tag{24}$$

$$k_{22}^2+2a_{22}\rho_rk_{22}-q_{22}\rho_r-2\rho_rk_{12}=0 \tag{25}$$

$$k_{11}=k_{12}k_{22}/\rho_r+a_{21}k_{22}+a_{22}k_{12}-q_{12} \tag{26}$$

Here q_{11}, q_{12}, and q_{22} are the elements of Q_r, and the sign ambiguities in taking the roots of Eqs. (24) and (25) are resolved by the conditions for positive definiteness of K_r; that is, $k_{11}>0$, $k_{22}>0$, and $k_{11}k_{22}-k_{12}^2>0$.

This generalization of the analytic Riccati solution for A_r in companion form, rather than the skew symmetric Eq. (10), requires modification of the form of the remaining equations of the original development. Specifically, the choice of state coordinates in Eq. (7) will be simply ν_r and $\dot{\nu}_r$, and similarly the partition of $W_r(t)$ in Eq. (9) becomes $[0\ f_r(t)]^T$. For flexible modes of the system, the nonzero elements of B_r are simply $\phi_r(p_i)$; for rigid body rotational modes, the nonzero elements of B_r are constants for torque actuators and linear functions of position for force actuators. With these changes, the generalization carries through to all subsequent results in the paper. We return to the restricted form expressed by Eqs. (7-13) for the remainder of the paper (with the exception of the example generated) simply to facilitate further discussion and comparison with the original development.

Remark 2: The physical significance of the weighting matrices R in Eq. (14) and R_{IMSC} in Eq. (17) are different. If R is diagonal, it weights the relative importance of equal magnitude inputs $u_i(t)$. By contrast, weights ρ_r of Eq. (22) give the relative importance of equal magnitudes of the generalized forces $f_r(t)/\omega_r$ and have no direct influence on the magnitude of the individual control inputs $u_i(t)$. In fact, the identical control solution can be obtained by absorbing the ρ_r into the corresponding Q_r in each J_r of Eq. (19), and replacing element $[R_r]_{22}$ with unity for all r. At this point it becomes clear that the only effect of ρ_r was to weight the relative importance of controlling the rth mode, but this is, in fact, the designer's objective in specifying Q_r.

Remark 3: Examination of the decoupled Riccati equations, Eq. (21), and their associated feedback control laws, Eq. (20), shows that the actuator location information, inherent in B alone, has no influence on the IMSC optimal control solution. The actuator locations can, therefore, be chosen based on any desired criterion, and this becomes an independent design step. This decoupling of the actuator placement problem and the optimal control problem is attractive computationally, but is not made without some loss of "optimality."

Remark 4: Since the optimal control solution for IMSC is found in the modal space, there exists an extra step of synthesizing this solution in terms of actual control inputs u. From Eqs. (5) and (11), $W(t)=Bu(t)$, but this expression cannot be inverted directly to obtain $u(t)$. The structure of W and B of Eqs. (8), (9), (12), and (13) leads to an equivalent relation

$$f(t)=B'u(t)=\begin{bmatrix} \phi_1(p_1)\cdots\phi_1(p_m) \\ \vdots \qquad \vdots \\ \phi_n(p_1)\cdots\phi_n(p_m)\end{bmatrix}u(t) \tag{27}$$

The solution of Eq. (27) for $u(t)$ in terms of $f(t)$ requires inversion of matrix B'. This leads to the fundamental limitation of IMSC: the requirement that the number of actuators equal the number of modes in the model $(m=n)$, a necessary condition for the existence of $(B')^{-1}$.

Remark 5: The fundamental property of IMSC is that generalized control functions $f(t)$ are designed for the system represented by Eq. (2). The method, therefore, is not limited to the linear-quadratic optimal control implementation. In Ref. 4, a pole placement design technique is developed in terms of IMSC, resulting in an attractive analytic control design.

In the next section, a technique is developed by which IMSC is enhanced by relaxing the stringent requirement on the number of actuators, through the use of a modified cost functional. Then, for the first time, a clear, concise explanation of the importance of actuator placement in IMSC is presented. This leads to the development of both open- and closed-loop approaches to actuator location optimization. Taking advantage of the decoupling noted above, these algorithms form an independent step in the control system design.

Reduction in Number of Actuators

The principal disadvantage of IMSC is the requirement that the number of actuators equal the number of modes in the control system model. This fact completely nullifies, for practical purposes, the advantage that extremely large-order Riccati equations can be handled by this design technique. In the design process there would also most likely be a management distaste for letting a computational tool dictate the hardware design (i.e., the number of actuators), and this is especially true since the hardware configuration is usually frozen in the design evolution long before the system software. Thus, this section is devoted to relaxing this stringent restriction on the number of actuators.

Adopting the point of view suggested in remark 2 that each ρ_r should be set to unity, the cost functional can be written in terms of W as

$$J'_{\mathrm{IMSC}}=\int_0^T (x^TQ'x+W^TR'W)\,\mathrm{d}t \tag{28}$$

$$Q'=\mathrm{diag}(Q_1/\rho_1,\ldots,Q_n/\rho_n)$$

$$R'=\mathrm{diag}(\infty,1,\ldots,\infty,1) \tag{29}$$

and the associated physical control $u(t)$ is to satisfy

$$W(t)=Bu(t) \tag{30}$$

When the number of actuators is less than the number of modes $(m<n)$, there will generically be no $u(t)$ that can produce the optimal $W(t)$.

Consider the least-square approximate solution to Eq. (30) given by

$$\bar{u}(t)=B\dagger W(t) \tag{31}$$

where

$$B\dagger=(B^TB)^{-1}B^T \tag{32}$$

[Reference 4 dismisses this use of a pseudoinverse, since it generically will not yield an exact realization of the generalized control $W(t)$.] Recalling Eq. (20), the feedback control which results from this approximate realization of $W(t)$ may be written

$$\bar{u}(t)=-B\dagger(R')^{-1}K'x(t)=-\bar{G}x(t) \tag{33}$$

where $K'=\text{block diag}(K'_1\ldots K'_n)$ is the compsoite matrix of solutions to the Riccati equations, Eqs. (21) with Q_r and R_r dictated by Eqs. (29).

It remains to characterize the nature of this feedback control law, which is suboptimal with respect to the original cost functional, Eq. (28). The task of identifying the performance index or set of indices for which a given feedback system is optimal is known as the inverse problem of linear optimal control.[5,6] Before applying the various techniques of the inverse problem to generate performance indices for which the feedback system Eqs. (11) and (33) is optimal, it will be useful to state several of the general results of the inverse problem.

Consider a general feedback system defined over the finite interval (t_0,t_1) by the equations

$$\dot{x}(t)=A(t)x(t)+B(t)u(t) \tag{34}$$

$$u(t)=-G(t)x(t) \tag{35}$$

Theorem: Every system which takes the form of Eqs. (34) and (35) is optimal with respect to some performance functional of the form

$$J=x^T(t_1)Fx(t_1)+\int_{t_0}^{t_1}[x^TQ(t)x+2u^TS(t)x+u^TR(t)u]\,\mathrm{d}t \tag{36}$$

where Q, R, and F are symmetric and R is positive definite. (The proof is a trivial matter and may be found in Ref. 5.)

From the standard linear-quadratic problem with a cross-term in the cost functional, $G(t)$ in Eq. (35) satisfies

$$G=R^{-1}(S+B^TK) \tag{37}$$

$$\dot{K}=-KA-A^TK+(KB+S^T)R^{-1}(S+B^TK)-Q \tag{38}$$

or

$$\dot{K}=-KA-A^TK+G^TRG-Q,$$

and

$$K(t_1)=F \tag{39}$$

It is interesting to note that the preceding theorem does not require that Q be positive semidefinite; hence, the feedback control, Eq. (35), may be destabilizing and yet still be optimal with respect to Eq. (36). Clearly for large space structure control design using our new formulation of IMSC, we desire the feedback control, Eq. (33), to be stabilizing; therefore, it must be verified independently that the closed-loop system $(A+B\bar{G})$ is stable.

The solution to the inverse problem is not in general unique, and there exist several methods for generating matrices Q, R, S, and F which represent solutions. Here we will present just two methods, drawn from Refs. 7 and 5, respectively, which separately address the infinite- and finite-time problems.

Method A: For the specific problem $(t_0, t_f) = (0, \infty)$ with A, B, and G invariant and $(A+BG)$ asymptotically stable, a solution of the form of Eq. (36) to the inverse problem Eqs. (34) and (35) may be obtained as follows:

1) Choose a symmetric positive-definite matrix R arbitrarily.

2) Choose a symmetric matrix K arbitrarily.

3) Derive S using Eq. (37); that is, $S=RG-B^TK$.

4) Derive Q from the algebraic form of Eq. (39), with $\dot{K}=0$. For the more general time-varying finite-terminal-time problem, method B is sufficient to generate a solution.

Method B:

1) Let G_0 be any matrix, equivalent in dimension to G, such that for all $t\epsilon[t_0,t_f]$

a) G_0B has m linearly independent real eigenvectors and nonpositive eigenvalues, and

b) rank G_0B = rank G_0 = rank B.

(Choosing $G_0=B^TP$ for any real symmetric $P<0$ will satisfy both a and b.)

2) Any pair of symmetric positive-definite matrices R and K which satisfy $G_0=-R^{-1}B^TK$ will lead to a solution of the inverse problem. (Choosing $R=I$, in particular, will yield $K=-P$.)

3) Q and F may then be obtained from Eq. (39).

4) Finally, from Eq. (37), S is given by $S=R(G+G_0)$.

Using either method A or B, we may now return to the problem at hand and construct, for any stabilizing feedback control law of the form of Eq. (33), a performance index of the form of Eq. (37) for which the control is optimal. In each of the infinite- and finite-time cases, one specific choice of matrices yields a particularly simple representation of the resulting matrices. Denoting by overbars this specific solution, we may proceed as follows.

Following method A, select $\bar{R}$, the control weighting matrix for our new cost functional, Eq. (36), as

$$\bar{R}=B^TR'B \tag{40}$$

Recalling Eq. (30), it is clear that this is the same control penalty that appears in the original cost functional. Next choose

$$\bar{K}=K' \tag{41}$$

Substituting Eqs. (33) and (40) into the expression in step 3, we obtain the cross-term

$$\bar{S}=(B^TR'B)B\dagger(R')^{-1}K'-B^TK' \tag{42}$$

This expression may be simplified by first noting that the special structure of R' and B leads to the identities

$$B^TR'=B^T(R')^{-1}=B^T \tag{43}$$

Using Eq. (43), and expanding the pseudoinverse according to Eq. (32), $\bar{S}$ becomes

$$\bar{S}=B^TB(B^TB)^{-1}B^T(R')^{-1}K'-B^TK'=0 \tag{44}$$

Finally, we generate our new state penalty $\bar{Q}$ according to step 4

$$\bar{Q}=-K'A-A^TK'+K'BB\dagger K' \tag{45}$$

where again we have used the identities in Eq. (43). Recalling that our original state penalty Q' satisfies

$$Q'=-K'A-A^TK'+K'(R')^{-1}K' \tag{46}$$

we may write

$$\bar{Q}=Q'-K'[(R')^{-1}-BB\dagger]K' \tag{47}$$

The feedback control law, Eq. (33), is, therefore, optimal with respect to a modified cost functional in which only the state weighting matrix has changed. In addition, we note that this particular solution to the inverse problem [defined by Eqs. (40), (44), and (47)] reduces to the original problem if $(B')^{-1}$ exists.

A similar result may be obtained using method B for the time-varying, finite-terminal-time problem. In step 1 we choose

$$G_0=-B\dagger K'(t) \tag{48}$$

where $K'(t)$ is the block-diagonal positive-definite solution to the differential Riccati equation of the original problem. It is simple to show that conditions a and b in step 1 are satisfied by Eq. (48). One of the possible solutions to $G_0=-(\bar{R})^{-1}B^T\bar{K}$ is then

$$\bar{R}=B^TB \tag{49}$$

$$\bar{K}=K' \tag{50}$$

Paralleling the procedure used in the infinite-time problem, we obtain the new state penalty $\bar{Q}$ in terms of the original penalty Q'

$$\bar{Q}=Q'-K'[(R')^{-1}-BB\dagger]K' \tag{51}$$

which is identical in form to Eq. (47), although $\bar{Q}$ is now a function of time. Further, we have $F=K'(T)=0$, and from step 4

$$\bar{S}=\bar{R}(\bar{G}+G_0)=0 \tag{52}$$

where the identities of Eq. (43) have been used one final time.

In general then, we may state that one performance index for which the feedback control Eq. (33) is optimal is

$$\bar{J}_{\rm IMSC}=\int_0^T(x^T\bar{Q}x+W^TR'W)\,dt \tag{53}$$

with $\bar{Q}$ defined by Eq. (51).

Remark 6: Reflecting on the form of Eq. (53) we recognize that the $\bar{u}$, given by Eq. (33), is that control among all possible realizable controls which minimizes a modified cost functional in which only the state penalty has changed. As in the original IMSC formulation, the control penalty term is entirely prespecified and the structure of the state penalty is restricted. Thus the limited design freedom available in the original formulation remains, since the diagonal blocks of Q' are still assignable by the designer. There is a loss of transparency in the design, however, since now the influence of the assignable elements of Q' on the state performance is not straightforward.

Remark 7: The method proposed here using $\bar{u}$ allows one to compute optimal control laws (relative to $\bar{J}_{\rm IMSC}$) for systems of arbitrarily large dimension, thus allowing all of the modal information one has about the system to be considered. Further, no specific requirement is placed on the number of actuators, although we must require that the feedback gain $\bar{G}$ stabilize the system. To within this constraint, there is no need for truncation in the control law design, since the computation of the Riccati equation solution from the analytical expressions, Eqs. (24-26), and the computation of the control

$\bar{u}$ (Eq. 31), requiring the inverse of an $m \times m$ matrix, can both be performed for extremely large systems.

Remark 8: The presence of both observation spillover and control spillover can destabilize the system, but neither one by itself can destabilize the system. Control spillover can be eliminated by the IMSC approach, provided a sufficient number of actuators can be used. And the modified method given here enhances this aspect of IMSC by relaxing this actuator constraint.

Remark 9: The design of optimal controllers using standard linear-quadratic theory is actually iterative. The designer chooses Q and R matrices, performs simulations to study the resulting performance, and iteratively adjusts the entries in R to prevent saturation of the actuators, and the entries in Q to obtain the desired performance. The advantages achieved by IMSC, in general, are made at the expense of freezing the choice of the control weighting matrix. One no longer has weighting factors with direct one-to-one correspondence to the input magnitudes $\bar{u}_i$, which, for conventional design, are adjusted to avoid individual actuator saturation. Only the entries in the Q matrix are at the designer's disposal, and these are properly intended for other purposes, and do not have a direct relation to any given actuator.

In a later section, actuator location optimization methods are developed which balance the system to cause the expectations of the commands to each actuator to be of similar magnitude. This can partially offset the lack of adjustment capability in the control penalty matrix.

Remark 10: The advantages of this new formulation of IMSC as described in remarks 7 and 8 are made at the expense of loss of design transparency as described in remark 6. However, considering the advantages inherent in lifting the stringent requirement on the number of actuators, a designer should be willing to pursue this new approach providing the resulting closed-loop system is stable. One sufficient condition for stability of the closed-loop system is that $\bar{Q}$ be positive-definite.

Table 1 Open-loop modal coefficients

Mode No.	$a_{21} \times 10^{-3}$	a_{22}
1	0.0	0.0
2	0.0	0.0
3	17.06	1.306
4	3.266	0.571
5	26.19	1.618
6	107.9	3.285
7	271.1	5.207
8	481.6	6.940
9	86.52	2.941
10	4.950	0.704
11	31.73	1.781

Numerical Example

The new augmented IMSC design approach for a reduced number of actuators demands careful consideration through application to a significant numerical example. The stability of the resulting suboptimal solution and the ability to iteratively improve the closed-loop system characteristics by varying the original cost functional are two primary concerns to be addressed.

The generic two-dimensional flexible spacecraft model developed by Hablani[8,9] is particularly attractive for this purpose. It is representative of shape-control problems, including both rigid and flexible motions, and the reduced models of the system are of moderate computational complexity. In particular, the reduced model used here is that described in Appendix C of Ref. 9, which includes 11 modes and six independent actuator commands. Here we will attempt to design a stable closed-loop control system for this model using IMSC without requiring either further model reduction or more actuators.

The system matrix A is block-diagonal with blocks in the form of Eq. (23). The numerical values of a_{21} and a_{22} for each of the 11 modes are summarized in Table 1. For modes 3 through 11, the values of a_{22} reflect the assumption of 0.5% modal damping. The control influence coefficient matrix B in the form of Eqs. (12) and (13) has the following nonzero rows:

−4.3086	−4.3086	−2.1543	0.0	0.0	0.0
0.0	0.0	0.0	1.7235	1.7235	0.8617
6.9267	−6.9267	0.0	6.2131	6.2131	6.5744
0.8839	−0.8839	0.0	7.5490	7.5490	−2.8571
−16.4660	16.4660	0.0	3.4908	3.4908	3.1422
2.1930	2.1930	−14.8935	−6.8832	6.8832	0.0
−28.3520	−28.3520	8.0206	−1.6243	1.6243	0.0
−23.5471	−23.5471	−8.8947	12.1737	−12.1737	0.0
−13.0526	13.0526	0.0	17.2428	17.2428	−2.3723
5.9554	5.9554	−3.0626	−4.0884	4.0884	0.0
−3.5566	−3.5566	−3.8992	13.7748	−13.7748	0.0

The six inputs are commands to ten (nonindependent) torque actuators grouped in five x-axis/y-axis pairs at the center and four corners of the structure. The details of the development of this model may be found in Ref. 9.

We will compare the optimal closed-loop system generated by the optimal feedback law, Eq. (20), with its suboptimal physical realization generated by the feedback law, Eq. (33), which is itself optimal with respect to Eq. (53). Note that the

Table 2 Open- and closed-loop system characteristics

Open-loop system		Optimal closed loop		Suboptimal closed loop	
Poles	ζ_i, %	Poles	ζ_i, %	Poles	ζ_i, %
$0 \pm 0j$	0	$-0.866 \pm 0.5j$	86.6	$-0.018 \pm 0.144j$	12.5
$0 \pm 0j$	0	$-0.866 \pm 0.5j$	86.6	$-0.028 \pm 0.178j$	15.6
$-0.653 \pm 131j$	0.5	$-0.822 \pm 131j$	0.630	$-0.819 \pm 131j$	0.627
$-0.286 \pm 57.1j$	0.5	$-0.576 \pm 57.1j$	1.01	$-0.386 \pm 57.1j$	0.675
$-0.809 \pm 162j$	0.5	$-0.951 \pm 162j$	0.588	$-0.929 \pm 162j$	0.574
$-1.64 \pm 329j$	0.5	$-1.72 \pm 329j$	0.523	$-1.70 \pm 329j$	0.519
$-2.60 \pm 521j$	0.5	$-2.65 \pm 521j$	0.509	$-2.64 \pm 521j$	0.508
$-3.47 \pm 694j$	0.5	$-3.51 \pm 694j$	0.505	$-3.50 \pm 694j$	0.504
$-1.47 \pm 294j$	0.5	$-1.55 \pm 294j$	0.528	$-1.54 \pm 294j$	0.523
$-0.352 \pm 70.4j$	0.5	$-0.611 \pm 70.4j$	0.869	$-0.372 \pm 70.4j$	0.529
$-0.891 \pm 178j$	0.5	$-1.02 \pm 178j$	0.573	$-0.959 \pm 178j$	0.538
Stability margin	0.0		0.576		0.018
Damping margin	0.0%		0.505%		0.504%

optimal system cannot be realized with the six actuator commands available but would require a minimum of 11 independent commands. We choose the weighting matrices Q' and R' according to Eq. (29) with $Q_i/\rho_i = 1$ for $i = 1,...,n$. The open-loop poles, and optimal and suboptimal closed-loop poles are compared in Table 2 along with the damping ratio ζ_i for each mode.

The suboptimal system is stable and each mode is more stable than in the open-loop system. The most significant degradation from the optimal closed-loop system occurs in the first two modes (which nonetheless remain well damped) and modes 4 and 10.

We may now choose among several desired objectives in tuning our suboptimal closed-loop design. One reasonable goal would be to re-establish the stability margin (the minimum among the magnitudes of the real parts of the poles) of the optimal modal-space design. Examining Table 2 indicates that the weights Q_i/ρ_i for modes 1, 2, 4, and 10 must be increased. After adjusting these quantities, examining the closed-loop properties, and adjusting once more, we find that $Q' = \text{diag}\ [10^4, 10^4, 10^4, 10^4, 1, 1, 10, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 10^2, 10^2, 1, 1]$ produces a stability margin of 0.719 for the physical realization of the closed-loop system.

A second objective in an iterative approach to control system design might be to improve the damping margin (the minimum among the damping ratios) of the resulting system. From Table 2, we see that the original choice of Q' has led to damping ratios that are frequency dependent; that is, the higher the frequency of a particular mode, the less the improvement in damping ratio. Again after two iterations we obtain for $Q' = \text{diag}\ [1, 1, 1, 1, 4, 4, 5, 5, 6, 6, 30, 30, 75, 75, 150, 150, 25, 25, 20, 20, 30, 30]$ a suboptimal damping margin of 0.714%. This choice of Q' reflects both an adjustment for the aforementioned frequency dependence and an adjustment for the degradation which results from the suboptimal realization.

For this numerical example we can draw two significant conclusions. First, in every iteration of the design, we found that not only was the suboptimal closed-loop system stable, but each mode was more stable than in the open-loop system. Second, it became clear that the process of iteratively adjusting Q' to achieve desired closed-loop system characteristics is straightforward. The suboptimal characteristics of a particular mode are directly influenced by changes in the corresponding diagonal elements of Q'.

Improvements can also be made in the control system design by a judicious choice of actuator locations. The optimal placement of actuators is considered in the next section.

Actuator Placement for IMSC

Baruh and Meirovitch[3] state that an advantage of IMSC is that the actuator locations are immaterial provided B' is not singular. The statement is valid in the sense that, as noted in remark 3, rather than solving for optimal control inputs, IMSC poses a problem which optimizes the generalized forces for each mode without regard for how these forces might be produced by the actuators. However, the determination of the actuator commands depends *fundamentally* on the actuator locations.

Remark 11: The consequences of improperly chosen actuator locations can be disastrous. It is a simple matter to generate examples which demonstrate that the force or torque required of a given a tuator can grow without bound as the actuator location approaches a position for which the system is uncontrollable.

We now develop methods for optimizing the actuator locations in order to minimize the actual control effort. First, a computationally simple open-loop approach is presented, and then, by similar reasoning, a more precise closed-loop approach is developed. Finally, a method is given which, in addition, minimizes residual mode excitation.

Open-Loop Approach to Actuator Placement

The objective of the actuator placement selection is to minimize the control effort $\bar{u}^T\bar{u}$. Let the singular value decomposition of $B\dagger$ be $U\Sigma\ V^T$ with singular values $\sigma_1(B\dagger) > 0,...,\sigma_m(B\dagger) > 0$, assuming that there are no redundant actuators. From Eq. (31),

$$\bar{u}^T\bar{u} = W^T(B\dagger)^T(B\dagger)W$$
$$= (W^TV)\text{diag}[\sigma_1^2(B\dagger),...,\sigma_m^2(B\dagger)](V^TW) \tag{54}$$

The matrix V is a unitary transformation that does not change the length of the vector, W. In the open-loop approach we assume no knowledge of the vectors W that will appear in operation, so that all values of W are equally likely; therefore, the control effort is minimized when the actuator locations p_j are chosen to make the largest $\sigma_i(B\dagger)$ as small as possible, or equivalently,

$$\max_{p_j}[\min_i\ \sigma_i(B)] \tag{55}$$

At this point, if a discrete set of possible actuator locations exists, the set chosen is simply the one which yields the largest minimum singular value. Alternatively, if a gradient search is to be performed over the space of possible actuator locations, the gradient of σ_i with respect to p_j may be developed as

$$\frac{\partial\sigma_i}{\partial p_j} = \eta_i^T\frac{\partial(B\dagger)^T(B\dagger)}{\partial p_j}\xi_i\Big/(\eta_i^T\xi_i)$$
$$= \eta_i^T\left(\frac{\partial B\dagger^T}{\partial p_j}B\dagger + B\dagger^T\frac{\partial B\dagger}{\partial p_j}\right)\xi_i\Big/(\eta_i^T\xi_i) \tag{56}$$

where ξ_i and η_i are the columns of V and U, respectively, and Eq. (56) is evaluated for i associated with the minimum singular value.

Closed-Loop Approach to Actuator Placement

Since W is to be generated as a feedback control, we can obtain information about its magnitude for an ensemble of initial conditions, and use this to improve upon the above technique. Consider the initial conditions to be unknown, and assume them to be zero mean and Gaussian with covariance $P_0 = E(x_0x_0^T)$. Then the actuators are to be placed to minimize the maximum eigenvalue of a control effort matrix

$$\mathcal{J} = E\int_0^T \bar{u}\bar{u}^T\text{d}t \tag{57}$$

for finite-time problems, and the limit as $T\to\infty$ otherwise.

With R' and K' defined as in Eqs. (29) and (33), the optimally controlled Eq. (5) and its solution are

$$\dot{x} = [A - (R')^{-1}K']x = \bar{A}x \tag{58}$$

$$x(t) = \Phi(t)x_0 \tag{59}$$

Recalling that $\bar{u}$ may be expressed as

$$\bar{u} = B\dagger W = -B\dagger(R')^{-1}K'x \tag{60}$$

the criterion $\mathcal{J}$ can be written as

$$\mathcal{J} = B\dagger\left\{(R')^{-1}\left[\int_0^T K'\Phi P_0\Phi^TK'\text{d}t\right](R')^{-1}\right\}B\dagger^T = B\dagger\mathcal{P}B\dagger^T \tag{61}$$

$$\mathcal{J} = B\dagger\mathcal{P}B\dagger^T \tag{62}$$

Note that because of the decoupling of the optimal control calculation from the actuator locations p_j, the optimal trajectory x and the Riccati equation solution K' are independent of p_j, making the matrix $\mathcal{P}$, defined in Eq. (62), independent of p_j. This fact vastly simplifies the computations required for optimizing the eigenvalues of $\mathcal{J}$ and makes the approach used here quite reasonable in terms of computational effort. The objective is to obtain

$$\min_{p_j}[\max_i \lambda_i(B\dagger \mathcal{P} B\dagger^T)] \tag{63}$$

which can be accomplished as before using gradient expressions for the eigenvalue derivatives.

In the infinite-time case, $T\rightarrow\infty$, the K' matrix becomes time-invariant. Letting $P(t)=\Phi(t)P_0\Phi^T(t)$, one can generate $\mathcal{P}$ from

$$\mathcal{P}=(R')^{-1}K'\left(\int_0^\infty P(t)\,\mathrm{d}t\right)K'(R')^{-1} \tag{64}$$

where $P(t)$ satisfies

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t}=\bar{A}P+P\bar{A} \tag{65}$$

Actuator Placement for Spillover Suppression

In the previous literature on IMSC, the number of modes to be controlled could not exceed the number of actuators, and control spillover questions were potentially quite serious. Reference 3, having adopted the attitude that actuator placement is immaterial for generating the control signals as long as the system is controllable, suggests placing the actuators at nodes of the first residual mode. Actually it is more reasonable to try to suppress the control spillover into a set of residual modes by placement of the actuators, but this of course would be done at the expense of the control effort required to control the controlled modes. Hence, a compromise would be called for. Let

$$\dot{x}_r=A_r x_r+B_r\bar{u} \tag{66}$$

represent the set of residual modes not included in the control system model. Then one wishes to keep $B_r\bar{u}=B_rB\dagger W$ small. In the open-loop approach, one would seek to adjust the actuator locations P_j to obtain

$$\max_{p_j}\ [\min_i \sigma_i(B)-\alpha_I\max_k \sigma_k(B_rB\dagger)] \tag{67}$$

where α_I is an adjustable parameter. In the closed-loop case, one wishes to keep

$$E\int_0^T B_r\bar{u}\bar{u}^T B_r^T\,\mathrm{d}t \tag{68}$$

small, which leads to the actuator placement criterion

$$\min_{p_j}[\max_i \lambda_i(B\dagger\mathcal{P}B\dagger^T)+\alpha_2\max_i \lambda_i(B_rB\dagger\mathcal{P}B\dagger^T B_r^T)] \tag{69}$$

The added complication of using such a criterion over that of Eq. (63) is negligible.

Conclusions

A new formulation of the independent modal space control method has been developed, which allows one to reduce the number of actuators required by this method. The analytical solution for the optimal control law is extended to handle modal damping and rigid body modes, and this allows one to solve the optimal control problem for very high dimensional systems. The price one pays for the ease of computation of the control law is threefold. The quadratic cost functional no longer contains the usual adjustable parameters which have a one-to-one correspondence to each actuator's control effort, and the state penalty in the cost functional only indirectly dictates the state performance. These two characteristics indicate that there can be more difficulty in tuning the cost functional to generate the desired performance. The third limitation is that although the approach allows one to consider a reduced number of actuators, stability of the closed-loop system must be assured. The design procedure then involves adjusting the cost functional parameters to meet stability margin requirements. A numerical example is given which shows that the approach is easy to use, and works on a significant satellite shape control problem of moderate complexity.

For the first time, actuator locations are given critical consideration within the framework of independent modal space control and new methods are developed for the optimal placement of actuators. The methods are computationally simple as a direct result of the modal space control framework and, in the original formulation, result in a decoupling of the control law determination and the actuator placement optimization.

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